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Frobenius submanifolds

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Abstract

The notion of a Frobenius submanifold — a submanifold of a Frobenius manifold which is itself a Frobenius manifold with respect to structures induced from the original Frobenius manifold — is studied. Two-dimensional submanifolds are particularly simple. More generally, sufficient conditions are given for a submanifold to be a so-called natural Frobenius submanifold. These ideas are illustrated using examples of Frobenius manifolds constructed from Coxeter groups, and for the Frobenius manifolds governing the quantum cohomology of $\mathbb{CP}^2$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$.

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1. Introduction

Substructures abound within mathematics. The purpose of this paper is to introduce the notion of a Frobenius submanifold — a submanifold of a Frobenius manifold which is itself a Frobenius manifold with respect to structures induced from the original Frobenius manifold. Certain specialized examples have appeared in the literature before, but the approach was more algebraic than geometric, the submanifolds being hyperplanes [12]. The paper is laid out as follows. In Section 2, a more general framework of induced substructures is given with Frobenius submanifolds being introduced in Section 3. The so-called natural Frobenius submanifolds are studied in more detail in Section 4, and in the remaining sections, a series of examples based on the foldings of Coxeter graphs and on the quantum cohomology of certain projective spaces are studied.

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2. Submanifolds and their induced structures

Let $\mathcal{M}$ be some manifold endowed with a metric $\eta = \langle\, ,\, \rangle$. Suppose further that on each tangent space $T_t\mathcal{M}$, one has a commutative multiplication of vectors

$$\circ : \quad T_t\mathcal{M} \times T_t\mathcal{M} \to T_t\mathcal{M}$$

varying smoothly over the manifold. Moreover, it will be assumed that this multiplication is compatible with the metric, in the sense that

$$\langle a \circ b, c\rangle = \langle a, b \circ c\rangle \quad \forall a, b, c \in T_t\mathcal{M}.$$

This property is known as the Frobenius condition. Let $\mathcal{F}$ denote the triple $\mathcal{F} = \{\mathcal{M}, \eta, \circ\}$. This will be called as a Frobenius structure.

Let $\mathcal{N} \subset \mathcal{M}$ be a submanifold of $\mathcal{M}$. One may define an induced $\mathcal{F}$ structure on $\mathcal{N}$, denoted by $\mathcal{F}_\mathcal{N} = \{\mathcal{N}, \eta_N, \star\}$, as follows. The metric $\eta_\mathcal{N} = \langle\, ,\, \rangle_\mathcal{N}$ is just the induced metric on $\mathcal{N}$, and $\star$ is defined by

$$a \star b = pr(a \circ b) \quad \forall a, b \in T_\tau\mathcal{N} \subset T_\tau\mathcal{M},$$

where pr denotes the projection (using the original metric η on $\mathcal{M}$) of $a \circ b \in T_\tau\mathcal{M}$ onto $T_\tau\mathcal{N}$ (Fig. 1). This induced multiplication may have very different algebraic properties than those of its progenitor.

Lemma 1. *The induced structure $\mathcal{F}_\mathcal{N}$ satisfies the Frobenius condition*

$$\langle a \star b, c\rangle_\mathcal{N} = \langle a, b \star c\rangle_\mathcal{N} \quad \forall a, b, c, \in T_\tau\mathcal{N}.$$

Hence $\mathcal{F}_\mathcal{N}$ is a Frobenius structure.

The proof follows immediately from the definitions. An alternative proof will be given below. Before this some general results will be given; this will also serve to fix the notation that will subsequently be used in this paper.

Let t^i, $i = 1, \ldots, m = dim\,\mathcal{M}$ be local coordinates on $\mathcal{M}$. With these, the submanifold $\mathcal{N}$ may be defined by the parametrization

$$t^i = t^i(\tau^\alpha), \quad \alpha = 1, \ldots, n = dim\,\mathcal{N}, \quad i = 1, \ldots, m = dim\,\mathcal{M}, \tag{1}$$

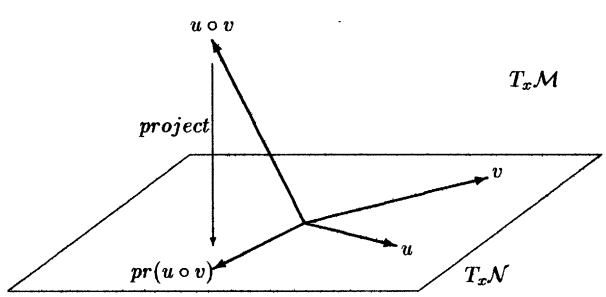


Fig. 1. The definition of the induced multiplication.

and so a basis for $T_\tau\mathcal{N}$ is given by

$$\frac{\partial}{\partial\tau^\alpha}=\frac{\partial t^i}{\partial\tau^\alpha}\frac{\partial}{\partial t^i}.$$

In these coordinates, the induced metric on $\mathcal{N}$ is given by [1]

$$\eta_{\alpha\beta}=\frac{\partial t^i}{\partial\tau^\alpha}\frac{\partial t^j}{\partial\tau^\beta}\eta_{ij}, \tag{2}$$

where η_{ij} is the metric on $\mathcal{M}$. The basis (1) may be extended to a basis for $T_t\mathcal{M}$, so

$$\frac{\partial}{\partial t^i}=A_i^\alpha\frac{\partial}{\partial\tau^\alpha}+n_i^{\tilde\alpha}\frac{\partial}{\partial\nu^{\tilde\alpha}}, \tag{3}$$

where $\tilde\alpha=1,\dots,m-n$ and

$$\frac{\partial}{\partial\nu^{\tilde\alpha}}\in(T_\tau\mathcal{N})^\perp.$$

Using the metrics on $T_t\mathcal{M}$ and $T_n\mathcal{N}$, one obtains

$$A_i^\alpha=\eta^{\alpha\beta}\eta_{ij}\frac{\partial t^j}{\partial\tau^\beta}.$$

The multiplication on $T_t\mathcal{M}$ may be defined in terms of a set of structure functions ${c_{ij}}^k(t^r)$:

$$\frac{\partial}{\partial t^i}\circ\frac{\partial}{\partial t^j}={c_{ij}}^k\frac{\partial}{\partial t^k}.$$

With these, one may find the induced structure functions for the multiplication on $T_\tau\mathcal{N}$.

$$\frac{\partial}{\partial\tau^\alpha}\circ\frac{\partial}{\partial\tau^\beta}=\frac{\partial t^i}{\partial\tau^\alpha}\frac{\partial t^j}{\partial\tau^\beta}\left.{c_{ij}}^k\right|_{\mathcal{N}}\frac{\partial}{\partial t^k}=\frac{\partial t^i}{\partial\tau^\alpha}\frac{\partial t^j}{\partial\tau^\beta}\left.{c_{ij}}^k\right|_{\mathcal{N}}\left[A_k^\gamma\frac{\partial}{\partial\tau^\gamma}+n_k^{\tilde\gamma}\frac{\partial}{\partial\nu^{\tilde\gamma}}\right].$$

Hence

$$\frac{\partial}{\partial\tau^\alpha}\star\frac{\partial}{\partial\tau^\beta}=pr\left[\frac{\partial}{\partial\tau^\alpha}\circ\frac{\partial}{\partial\tau^\beta}\right]=\frac{\partial t^i}{\partial\tau^\alpha}\frac{\partial t^j}{\partial\tau^\beta}\left.{c_{ij}}^k\right|_{\mathcal{N}}A_k^\gamma\frac{\partial}{\partial\tau^\gamma}={c_{\alpha\beta}}^\gamma\frac{\partial}{\partial\tau^\gamma},$$

where the induced structure functions are given by

$${c_{\alpha\beta}}^\gamma=\frac{\partial t^i}{\partial\tau^\alpha}\frac{\partial t^j}{\partial\tau^\beta}\frac{\partial t^r}{\partial\tau^\delta}\eta_{kr}\eta^{\gamma\delta}\left.{c_{ij}}^k\right|_{\mathcal{N}}. \tag{4}$$

Proof. With the notion set up the proof of the proposition is straightforward. The Frobenius property on $\mathcal{M}$ is equivalent to the condition that the tensor

$$c_{ijk}=\eta_{kl}{c_{ij}}^l$$

[1] The symbol η will be used to denote a metric on either $\mathcal{M}$ or $\mathcal{N}$ with Greek indices denoting structures on $\mathcal{N}$ and Latin indices denoting structures on $\mathcal{M}$. This convention will be used throughout this paper.

is totally symmetric (recall that $\circ$ is, by definition, commutative). It follows from this and (4) that

$$c_{\alpha\beta\gamma} = \frac{\partial t^i}{\partial \tau^\alpha}\frac{\partial t^j}{\partial \tau^\beta}\frac{\partial t^k}{\partial \tau^\gamma}\, c_{ijk}\big|_{\mathcal{N}} \tag{5}$$

is also totally symmetric. Hence, the induced structure $\mathcal{F}_{\mathcal{N}}$ inherits the Frobenius property. □

Example. Consider the Jordan algebra defined by the

$$e_1 \circ e_i = +e_i, \quad i = 1, \ldots, m, \qquad e_i \circ e_i = -e_1,$$
$$i = 2, \ldots, m, \qquad e_i \circ e_j = 0 \quad \text{otherwise}.$$

One may show that with the inner product defined by $\eta_{ij} = c_{ij}{}^k c_{kl}{}^l$ (where $c_{ij}{}^k$ are the structure constants of this algebra) this algebra has the Frobenius property [10]. These may then be used to define a trivial $\mathcal{F}$-structure — trivial in the sense that the structures do not vary are the tangent space varies. The above lemma may then be used to find examples of other, non-trivial, $\mathcal{F}$-structures.

In what follows, the idea of a natural substructure will be important.

Definition. A substructure $\mathcal{F}_{\mathcal{N}}$ of a Frobenius structure $\mathcal{F}$ is said to be *natural* if

$$a \star b = a \circ b \quad \forall a, b \in T_\tau\mathcal{N},$$

i.e. no projection onto $T_\tau\mathcal{N}$ is required, for all points $x \in \mathcal{N}$.

In terms of the local coordinates, this means that the $\frac{1}{2}n(n+1)(m-n)$ conditions $\Xi_{\alpha\beta}{}^{\tilde{\gamma}}$ must vanish, where

$$\Xi_{\alpha\beta}{}^{\tilde{\gamma}} = \frac{\partial t^i}{\partial \tau^\alpha}\frac{\partial t^j}{\partial \tau^\beta}\, c_{ij}{}^k\Big|_{\mathcal{N}} n_k{}^{\tilde{\gamma}}. \tag{6}$$

Example. Let $I \subset \{1, 2, \ldots, m\}$ and suppose that $\mathcal{N}$ is given by the conditions $t^i = 0$ for $i \notin I$. Then the obstruction reduces to the algebraic condition

$$c_{ij}{}^k\Big|_{\mathcal{N}} = 0, \quad i, j \in I, \ \ k \notin I.$$

This condition was derived in [12] in the context of Frobenius manifolds constructed from Coxeter groups (see Section 5). Here it is a specialization of the more general condition (6).

3. Frobenius manifolds

One particular class of Frobenius structures are Frobenius manifolds. A Frobenius manifold may be defined as follows [2]. Let $F = F(t^i)$ be a function — the prepotential — defined on some region $\mathcal{M} \subset \mathbb{R}^m$ (sometimes $\mathcal{M} \subset \mathbb{C}^m$) such that the third derivatives

$$c_{ijk} = \frac{\partial^3 F}{\partial t^i \partial t^j \partial t^k}$$

satisfy the following conditions:

- Normalization:

  $$\eta_{ij} = c_{1ij}$$

  is a constant, non-degenerate matrix. Let $\eta^{ij} = (\eta_{ij})^{-1}$. These may be used to raise and lower indices.
- Associativity: the functions

  $$c_{ij}{}^{k} = \eta^{kl} c_{ijl}$$

  define an associative, commutative algebra

  $$\frac{\partial}{\partial t^i} \circ \frac{\partial}{\partial t^j} = c_{ij}{}^{k} \frac{\partial}{\partial t^k}$$

  on each tangent space $T_t\mathcal{M}$ with unity element $\mathbb{I}$, so $\mathbb{I} \circ a = a\, \forall a \in T_t\mathcal{M}$. The above normalization implies that $\mathbb{I} = \partial_{t^1}$. The resulting differential equation for the prepotential is known as the Witten–Dijkgraaf–Verlinde–Verlinde (or WDVV) equation.
- Homogeneity: the function F must be quasi-homogeneous, so

  $$\mathcal{L}_E = d_F F + \text{quadratic terms},$$

  where $\mathcal{L}_E$ is the Lie derivative along the Euler vector field

  $$E = (q^i_j t^j + r^i) \frac{\partial}{\partial t^i}.$$

The most common form (which is canonical under certain additional requirements) for η_{ij} is the antidiagonal form

$$\eta_{ij} = \delta_{i+j,m+1},$$

and this form will be assumed throughout this paper. It then follows from the above axioms that the prepotential takes the general form [2]

$$F = \frac{1}{2} t_1^2 t_m + \frac{1}{2} t_1 \sum_{j=2}^{m-1} t_i t_{m-i+1} + f(t_2, \ldots, t_m). \tag{7}$$

It will be assumed that the Euler vector field E takes the form

$$E = \sum_i d_i t^i \frac{\partial}{\partial t^i} + \sum_{i\,|\,d_i=0} r^i \frac{\partial}{\partial t^i}$$

with $d_1 = 1$, and with the canonical form (7) for the prepotential

$$q_i + q_{m+1-i} = d,$$

where $d = 3 - d_F$ and $d_i = 1 - q_i$.

[2] To avoid a plethora of brackets in terms such as $(t^2)^2(t^3)^3$ the indices on t will be written downstairs in *explicit* formulae, so $t_i = t^i$, not $t_i = \eta_{ij} t^j$.

Example ($m = 2$). The equations of associativity are vacuous, so any function

$$F = \tfrac{1}{2}t_1 t_2^2 + f(t_2)$$

defines a Frobenius manifold. If the quasi-homogeneity condition is now used, the otherwise free function $f(t_2)$ is constrained to take one of the five forms.

Example ($m = 3$). The equations of associativity results in a single differential equation for $f(x, y)$,

$$f_{xxy}^2 = f_{yyy} + f_{xxx} f_{xyy}.$$

If the quasi-homogeneity condition is now used, this equation may be reduced to various third order ordinary differential equation, each equivalent to a Painlevé VI equation.

On a submanifold $\mathcal{N}$ one may, as well as the induced $\mathcal{F}_{\mathcal{N}}$ structures, define an induced vector field

$$E_{\mathcal{N}} = pr\, E|_{\mathcal{N}}.$$

This raises a number of questions on whether the induced structures are quasi-homogeneous with respect to the induced Euler vector field, and in particular:

- For what families of submanifolds does
  $$E_{\mathcal{N}} = (q_\beta^\alpha \tau^\beta + r^\alpha)\frac{\partial}{\partial \tau^\alpha},$$
  since in general $E_{\mathcal{N}}$ will not be linear in τ^i?
- For what families of submanifolds does
  $$E_{\mathcal{N}} = E|_{\mathcal{N}},$$
  or equivalently, $(E|_{\mathcal{N}})^\perp = 0$?

It will be shown in Section 3 that for natural Frobenius submanifolds, the second condition implies the first.

Definition. Let $\mathcal{F}$ be a Frobenius manifold. A submanifold $\mathcal{N}$ be said to be a Frobenius submanifold if $\mathcal{F}_{\mathcal{N}}$ is a Frobenius manifold with respect to the induced structures.

Definition. A natural Frobenius submanifold $\mathcal{N}$ is a Frobenius submanifold where

$$a \star b = a \circ b \quad \forall a, b \in T_\tau \mathcal{N},$$

or equivalently, $((a \star b)|_{\mathcal{N}})^\perp = 0$.

For the rest of this section, the quasi-homogeneity condition will be ignored, concentrating instead on properties of the induced multiplication on two-dimensional submanifolds. It turns out that two-dimensional Frobenius submanifolds are particularly simple.

Proposition 2. *Let $\mathcal{F} = \{\mathcal{M}, \eta, \circ\}$ be a Frobenius manifold and let $\mathcal{N}$ be a two-dimensional submanifold such the unity vector field at all points of $\mathcal{N}$ is always tangent to $\mathcal{N}$. Then $\mathcal{F}_{\mathcal{N}}$ is a Frobenius manifold.*

Proof. The proof will only be given for $dim\,\mathcal{M} = 3$, the general case being a direct generalization of the lower dimensional result. To fulfil the tangential condition, the surface may be parametrized by

$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tau_1 + \begin{pmatrix} a(\tau_2) \\ b(\tau_2) \\ c(\tau_2) \end{pmatrix},$$

this ensuring that $\partial_{t_1} = \partial_{\tau_1}$. The induced metric on the ruled surface is automatically flat, and flat coordinates are given by

$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tau_1 + \begin{pmatrix} -\frac{1}{2}\int b_{\tau_2}^2\,\mathrm{d}\tau_2 \\ b(\tau_2) \\ \tau_2 \end{pmatrix},$$

in which the induced metric is just $\eta|_{\mathcal{N}} = 2\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2$.

In this parametrization,

$$\frac{\partial}{\partial \tau_1} = +\frac{\partial}{\partial t_1}, \qquad \frac{\partial}{\partial \tau_2} = -\frac{1}{2}\frac{\partial}{\partial t_1} + b_{\tau_2}\frac{\partial}{\partial t_2} + \frac{\partial}{\partial t_3}.$$

Since ∂_{t_1} is the unity vector field,

$$\frac{\partial}{\partial \tau_1} \star \frac{\partial}{\partial \tau_1} = \frac{\partial}{\partial \tau_1}, \qquad \frac{\partial}{\partial \tau_1} \star \frac{\partial}{\partial \tau_2} = \frac{\partial}{\partial \tau_1},$$

and so it just remains to calculate $\partial_{\tau_2} \circ \partial_{\tau_2}$ and project this onto $T_x\mathcal{N}$. The vector

$$\frac{\partial}{\partial \nu} = \frac{\partial}{\partial t_2} - b_{\tau_2}\frac{\partial}{\partial \tau_1}$$

is perpendicular to $T_\tau\mathcal{N}$ (it is not necessary to normalize here) and hence

$$\frac{\partial}{\partial \tau_2} \circ \frac{\partial}{\partial \tau_2} = \left(-\frac{3}{4}b_{\tau_2}^4 + b_{\tau_2}^3 c_{222}|_{\mathcal{N}} + 3b_{\tau_2}^2 c_{223}|_{\mathcal{N}} + 3b_{\tau_2} c_{233}|_{\mathcal{N}} + c_{333}|_{\mathcal{N}}\right)\frac{\partial}{\partial \tau_1}$$
$$+(-b_{\tau_2}^3 + b_{\tau_2}^2 c_{222}|_{\mathcal{N}} + 2b_{\tau_2} c_{223}|_{\mathcal{N}} + c_{233}|_{\mathcal{N}})\frac{\partial}{\partial \nu}.$$

Hence, on projecting onto $T_\tau\mathcal{N}$,

$$\frac{\partial}{\partial \tau_2} \star \frac{\partial}{\partial \tau_2} = pr\left(\frac{\partial}{\partial \tau_2} \circ \frac{\partial}{\partial \tau_2}\right) = [\text{function of } \tau_2]\frac{\partial}{\partial \tau_1}.$$

If $dim\,\mathcal{N}$ was greater than 2, one would now have to check that this multiplication was associative, but in two dimensions, the associativity condition is vacuous, and this induced structure is automatically a Frobenius submanifold with prepotential

$$F_{\mathcal{N}} = \frac{1}{2}\tau_1^2\tau_2 + \iiint [\text{function of } \tau_2]\,\mathrm{d}\tau_2\,\mathrm{d}\tau_2\,\mathrm{d}\tau_2.$$

The condition for the surface to be a natural Frobenius submanifold is thus

$$b^3_{\tau_2} = b^2_{\tau_2} c_{222}|_{\mathcal{N}} + 2b_{\tau_2} c_{223}|_{\mathcal{N}} + c_{233}|_{\mathcal{N}},$$

a first order ordinary differential equation of degree 3. Note that in general

$$F|_{\mathcal{N}} \neq F_{\mathcal{N}}.$$

Thus any two-dimensional manifold ruled in this way is a Frobenius submanifold. □

4. Natural Frobenius submanifolds

In this section, sufficient conditions will be derived to ensure that a flat submanifold of a Frobenius manifold is a natural Frobenius submanifold.

Theorem 3. *Let $\mathcal{N}$ be a flat submanifold of a Frobenius manifold $\mathcal{M}$ with*

$$(\mathbb{I}|_{\mathcal{N}})^{\perp} = 0, \qquad ((a \circ b)|_{\mathcal{N}})^{\perp} = 0 \quad \forall a, b \in T_{\tau}\mathcal{N}, \qquad (E|_{\mathcal{N}})^{\perp} = 0.$$

Then $\mathcal{N}$ is a natural Frobenius submanifold.

With so many conditions on $\mathcal{N}$, the result might seem inevitable, but it is not clear that a prepotential exists, or that the induced Euler vector field is linear, or that the induced prepotential is quasi-homogeneous with respect to the induced Euler vector field.

Proof. Since $\mathcal{N}$ is flat, one may, by solving the Gauss–Manin equations, find coordinates so that the components of the induced metric (2) are constant — the so-called flat coordinates. The geometric properties of a flat submanifold in a flat manifold is summarized in Appendix A. □

Existence of induced prepotential. Since $a \circ b = a \star b$, it follows immediately that $\circ$ is a commutative, associative multiplication with induced structure functions given by (4). The existence of an induced prepotential $F_{\mathcal{N}}$ such that

$$c_{\alpha\beta\gamma} = \frac{\partial^3 F_{\mathcal{N}}}{\partial\tau^{\alpha}\partial\tau^{\beta}\partial\tau^{\gamma}}$$

is given by the integrability conditions

$$\frac{\partial c_{\alpha\mu\nu}}{\partial\tau^{\beta}} - \frac{\partial c_{\beta\mu\nu}}{\partial\tau^{\alpha}} = 0.$$

Using (5) and (A.4)

$$\begin{aligned}\frac{\partial c_{\alpha\mu\nu}}{\partial\tau^{\beta}} - \frac{\partial c_{\beta\mu\nu}}{\partial\tau^{\alpha}} &= \sum_{\text{similar terms}} \pm \frac{\partial t^i}{\partial\tau^{\sigma}} \frac{\partial^2 t^j}{\partial\tau^{\beta}\partial\tau^{\mu}} \frac{\partial t^k}{\partial\tau^{\nu}} \, c_{ijk}\big|_{\mathcal{N}} \\ &= \sum_{\text{similar terms}} \pm \frac{\partial t^i}{\partial\tau^{\sigma}} \frac{\partial t^k}{\partial\tau^{\nu}} \varOmega_{\beta\mu}{}^{\tilde{\alpha}} n_{\tilde{\alpha}}{}^{j} \, c_{ijk}\big|_{\mathcal{N}} = \sum_{\text{similar terms}} \pm \varOmega_{\beta\mu}{}^{\tilde{\alpha}} \varXi_{\tilde{\alpha}\alpha\nu}.\end{aligned}$$

The obstruction to the existence of a prepotential is thus

$$\text{obstruction} = \Omega^{[\alpha}_{(\mu}\Xi^{\beta]}_{\nu)}$$

(suppressing the sum over $\tilde{\alpha}$). Two simple cases where this obstruction vanishes are:

$$\Xi = 0, \qquad \Xi = \Omega.$$

Thus, on a natural submanifold these obstructions vanish (since Ξ vanishes) and an induced prepotential $F_{\mathcal{N}}$ exists. Proposition 2 shows that this condition is not necessary.

Existence of unity element. Recall that

$$\mathbb{I} = \frac{\partial}{\partial t^1}.$$

Using this and (3), it follows that $n_1^{\tilde{\alpha}} = 0$, and this together with (A.4) implies that

$$\frac{\partial^2 t^m}{\partial \tau^\alpha \partial \tau^\beta} = 0, \quad m = \dim \mathcal{M}.$$

Hence $t^m = \mu_\alpha \tau^\alpha + \beta$, where μ_α and β are constants. Linear transformations, which do not affect the flatness of the τ-coordinates, may be used to fix $t^m = \tau^n$. This ensures that

$$\mathbb{I}_{\mathcal{N}} = pr(\mathbb{I}|_{\mathcal{N}}) = \frac{\partial t^m}{\partial \tau^\alpha}\eta^{\alpha\beta}\frac{\partial}{\partial \tau^\beta} = \frac{\partial}{\partial \tau^1}.$$

The parametrization of the submanifold must have the generic form

$$\begin{aligned} &t^1 = \tau^1 + f^1(\tau^2, \ldots, \tau^n), \qquad t^i = f^i(\tau^2, \ldots, \tau^n), \\ &i = 2, \ldots, m-1, \qquad t^m = \tau^n. \end{aligned} \tag{8}$$

Having set up appropriate coordinates on $\mathcal{N}$, the required normalization on the submanifold is straightforward:

$$c_{1\alpha\beta} = \frac{\partial t^i}{\partial \tau^1}\frac{\partial t^j}{\partial \tau^\alpha}\frac{\partial t^k}{\partial \tau^\beta}\, c_{ijk}\big|_{\mathcal{N}} = c_{1jk}|_{\mathcal{N}}\frac{\partial t^j}{\partial \tau^\alpha}\frac{\partial t^k}{\partial \tau^\beta} = \eta_{jk}\frac{\partial t^j}{\partial \tau^\alpha}\frac{\partial t^k}{\partial \tau^\beta} = \eta_{\alpha\beta}.$$

Linearity of induced Euler vector field. Let

$$E = E^i\frac{\partial}{\partial t^i}.$$

Then, on using (3),

$$E|_{\mathcal{N}} = E^i\left\{A_i^\alpha\frac{\partial}{\partial \tau^\alpha} + n_i^{\tilde{\alpha}}\frac{\partial}{\partial \nu^{\tilde{\alpha}}}\right\}.$$

Thus, if $(E|_{\mathcal{N}})^\perp = 0$,

$$E^i|_{\mathcal{N}} n_i{}^{\tilde{\alpha}} = 0, \tag{9}$$

$$E^\alpha_{\mathcal{N}} = E^i|_{\mathcal{N}}\eta_{ij}\frac{\partial t^j}{\partial \tau^\beta}\eta^{\alpha\beta}, \tag{10}$$

It follows from (9) that

$$E^i|_{\mathcal{N}} = \omega^\alpha \frac{\partial t^i}{\partial \tau^\alpha}$$

for some function $\omega^\alpha(\tau)$ and with this (10) implies $\omega^\alpha = E^\alpha_{\mathcal{N}}$. Thus

$$E^i|_{\mathcal{N}} = E^\alpha_{\mathcal{N}} \frac{\partial t^i}{\partial \tau^\alpha}. \tag{11}$$

To prove that $E_{\mathcal{N}}$ is linear in τ, its second derivatives will be calculated. From (10),

$$\frac{\partial E^\alpha_{\mathcal{N}}}{\partial \tau^\sigma} = \frac{\partial t^k}{\partial \tau^\sigma} \left.\frac{\partial E^i}{\partial t^k}\right|_{\mathcal{N}} \eta_{ij} \frac{\partial t^j}{\partial \tau^\beta} \eta^{\alpha\beta} + E^i|_{\mathcal{N}} \eta_{ij} \frac{\partial^2 t^j}{\partial \tau^\sigma \tau^\beta} \eta^{\alpha\beta}.$$

Using (A.4) and (9), the second term vanishes. Thus

$$\frac{\partial^2 E^\alpha_{\mathcal{N}}}{\partial \tau^\sigma \partial \tau^\nu} = \eta_{ij} \frac{\partial^2 t^j}{\partial \tau^\beta \partial \tau^\nu} \eta^{\alpha\beta} \frac{\partial t^k}{\partial \tau^\sigma} \left.\frac{\partial E^i}{\partial t^k}\right|_{\mathcal{N}} + \eta_{ij} \frac{\partial t^j}{\partial \tau^\beta} \eta^{\alpha\beta} \frac{\partial^2 t^k}{\partial \tau^\sigma \partial \tau^\nu} \left.\frac{\partial E^i}{\partial t^k}\right|_{\mathcal{N}}, \tag{12}$$

using the fact that E^i is linear in t. The first term in (12) simplifies on using (A.4):

$$\text{first term} = \eta_{ij} \Omega_{\beta\nu}{}^{\tilde{\alpha}} n_{\tilde{\alpha}}{}^j \eta^{\alpha\beta} \frac{\partial t^k}{\partial \tau^\sigma} \left.\frac{\partial E^i}{\partial t^k}\right|_{\mathcal{N}}.$$

This is simplified by first differentiating (9) and using (A.6), yielding

$$\text{first term} = \eta^{\alpha\beta} \Omega_{\sigma\delta}{}^{\tilde{\alpha}} \Omega_{\tilde{\alpha}\beta\nu} E^\delta_{\mathcal{N}}.$$

The second term in (12) may be written, using the explicit form $E^i = q^i_j t^j + r^i$ and (A.4) as

$$\text{second term} = \eta_{ks} \frac{\partial}{\partial \tau^\beta} \{\eta_{ij} q^i_r \eta^{rs} t^j\} \eta^{\alpha\beta} \Omega^{\tilde{\alpha}}_{\sigma\nu} n^k_{\tilde{\alpha}}.$$

Using the explicit form $q^i_j = (1 - q_i)\delta_{ij}$ with $q_i + q_{m+1-i} = d$,

$$\eta_{ij} q^i_r \eta^{rs} = -q^s_j + (2 - d)\delta^s_j.$$

Hence

$$\text{second term} = \left\{ -\eta_{ks} \frac{\partial E^s}{\partial \tau^\beta} + (2 - d)\eta_{ks} \frac{\partial t^s}{\partial \tau^\beta} \right\} \eta^{\alpha\beta} \Omega^{\tilde{\alpha}}_{\sigma\nu}.$$

Repeating the earlier manipulations gives

$$\frac{\partial^2 E^\alpha_{\mathcal{N}}}{\partial \tau^\sigma \partial \tau^\nu} = \eta^{\alpha\beta} \eta^{\tilde{\alpha}\tilde{\beta}} \{\Omega^{\tilde{\alpha}}_{\sigma\delta} \Omega^{\tilde{\beta}}_{\beta\nu} - \Omega^{\tilde{\alpha}}_{\beta\delta} \Omega^{\tilde{\beta}}_{\sigma\nu}\} E^\delta_{\mathcal{N}},$$

and by virtue of the Gauss–Codazzi equation (A.7) this vanishes. Hence $E_{\mathcal{N}}$ is linear in the τ-variables.

Quasi-homogeneity of induced prepotential. The prepotential F satisfies the quasi-homogeneity condition

$$\mathcal{L}_E F = d_F F + \text{quadratic terms}.$$

This is equivalent to the relation

$$\mathcal{L}_E c_{ijk} = d_F c_{ijk}$$

on structure functions. Expanding this gives

$$E^r \frac{\partial c_{ijk}}{\partial t^r} = d_F c_{ijk} - \frac{\partial E^r}{\partial t^i} c_{rjk} - \text{cyclic}.$$

Since the induced prepotential on $\mathcal{N}$ is only defined implicitly, the analogous relation for the quasi-homogeneity of the induced structure functions with respect to the induced vector field will be found, the quasi-homogeneity following by integration of this result. The proof is straightforward:

$$E_{\mathcal{N}} c_{\alpha\beta\gamma} = E^{\sigma}_{\mathcal{N}} \frac{\partial c_{\alpha\beta\gamma}}{\partial \tau^{\sigma}} = E^{\sigma}_{\mathcal{N}} \frac{\partial}{\partial \tau^{\sigma}} \left\{ \frac{\partial t^i}{\partial \tau^{\alpha}} \frac{\partial t^j}{\partial \tau^{\beta}} \frac{\partial t^k}{\partial \tau^{\gamma}} c_{ijk}|_{\mathcal{N}} \right\}.$$

But terms like

$$E^{\sigma}_{\mathcal{N}} \frac{\partial^2 t^i}{\partial \tau^{\sigma} \partial \tau^{\beta}} \frac{\partial t^j}{\partial \tau^{\beta}} \frac{\partial t^k}{\partial \tau^{\gamma}} c_{ijk}|_{\mathcal{N}} = E^{\sigma}_{\mathcal{N}} \Omega_{\sigma\alpha}{}^{\tilde{\mu}} \Xi_{\beta\gamma\tilde{\mu}}$$

vanish since $\mathcal{N}$ is a natural submanifold. Thus

$$E_{\mathcal{N}} c_{\alpha\beta\gamma} = \frac{\partial t^i}{\partial \tau^{\alpha}} \frac{\partial t^j}{\partial \tau^{\beta}} \frac{\partial t^k}{\partial \tau^{\gamma}} E^{\sigma}_{\mathcal{N}} \frac{\partial t^p}{\partial \tau^{\sigma}} \left. \frac{\partial c_{ijk}}{\partial t^p} \right|_{\mathcal{N}}.$$

Using (11) and the quasi-homogeneity of F gives

$$E_{\mathcal{N}}(c_{\alpha\beta\gamma}) = d_F c_{\alpha\beta\gamma} - \left\{ \frac{\partial E^{\sigma}}{\partial \tau^{\alpha}} c_{\sigma\beta\gamma} + E^{\sigma}_{\mathcal{N}} \Omega_{\alpha\sigma}{}^{\tilde{\alpha}} \Xi_{\beta\gamma\tilde{\alpha}} \right\} - \text{cyclic}.$$

Hence on a natural submanifold

$$\mathcal{L}_{E_{\mathcal{N}}} c_{\alpha\beta\gamma} = d_F c_{\alpha\beta\gamma},$$

where $\mathcal{L}_{E_{\mathcal{N}}}$ is the Lie derivative along $E_{\mathcal{N}}$ in the submanifold $\mathcal{N}$. Integration then gives the quasi-homogeneity of the induced prepotential. Note that the total scaling dimension d_F is unchanged. This result is actually independent of the condition $\Xi = 0$, the terms involving Ξ cancel. Thus, on any submanifold where $(E_{\mathcal{N}})^{\perp} = 0$ the induced structure functions of the not necessarily associative induced algebra are quasi-homogeneous. This result may be formulated in terms of the vanishing of the induced Dubrovin connection [2].

4.1. *The induced intersection form*

One important property of a Frobenius manifold is the existence of a second flat metric defined by [2]

$$g^{ij} = E(\mathrm{d}t^i \circ \mathrm{d}t^j) = c^{ij}_k E(\mathrm{d}t^k)$$

with the basic property that

$$\frac{\partial g^{ij}}{\partial t^1} = \eta^{ij}.$$

It follows from this that the pencil of metrics

$$g_\lambda^{ij} = g^{ij} + \lambda \eta^{ij}$$

is flat for all values of λ. In this section, it will be shown (under the conditions of the above theorem) that the restriction of this metric to the submanifold is given by the analogous formulae. Since the above defines g^{ij} rather than g_{ij}, a different approach is required.

Consider the tensor

$$g^{ij} \frac{\partial}{\partial t^i} \otimes \frac{\partial}{\partial t^j}.$$

Restricting this to $\mathcal{N}$, and using (3) gives

$$\begin{aligned} g^{ij} \frac{\partial}{\partial t^i} \otimes \frac{\partial}{\partial t^j}\bigg|_{\mathcal{N}} &= g^{ij}|_{\mathcal{N}} \left\{ A_i{}^\alpha \frac{\partial}{\partial \tau^\alpha} + n_i{}^{\tilde\alpha} \frac{\partial}{\partial \tau^{\tilde\alpha}} \right\} \otimes \left\{ A_j{}^\beta \frac{\partial}{\partial \tau^\beta} + n_j{}^{\tilde\beta} \frac{\partial}{\partial \tau^{\tilde\beta}} \right\} \\ &= g^{ij}|_{\mathcal{N}} A_i{}^\alpha A_j{}^\beta \frac{\partial}{\partial \tau^\alpha} \otimes \frac{\partial}{\partial \tau^\beta} + g^{ij}|_{\mathcal{N}} n_i{}^{\tilde\alpha} n_j{}^{\tilde\beta} \frac{\partial}{\partial \tau^{\tilde\alpha}} \otimes \frac{\partial}{\partial \tau^{\tilde\beta}} \\ &\quad + 2 g^{ij}|_{\mathcal{N}} A_i{}^{\tilde\alpha} n_j{}^{\tilde\beta} \frac{\partial}{\partial \tau^\alpha} \otimes \frac{\partial}{\partial \nu^{\tilde\beta}}. \end{aligned}$$

Simple calculations show that, under the conditions of the above theorem,

$$\text{cross term} = 2\eta^{\alpha\beta} \Xi_{\beta\sigma}{}^{\tilde\beta} E^\sigma \frac{\partial}{\partial \tau^\alpha} \otimes \frac{\partial}{\partial \nu^{\tilde\beta}},$$

and hence vanish. This gives an orthogonal decomposition and hence a metric on $\mathcal{N}$ given by

$$g^{\alpha\beta} = g^{ij}|_{\mathcal{N}} A_i{}^\alpha A_j{}^\beta.$$

Similar manipulations give

$$g^{\alpha\beta} = E_{\mathcal{N}}(\mathrm{d}\tau^\alpha \star \mathrm{d}\tau^\beta).$$

Thus, the two ways to compute the induced intersection form, either by the restriction of the intersection from $\mathcal{M}$ to $\mathcal{N}$, or by calculating it using the induced Euler vector field on $\mathcal{N}$ agree. Similarly,

$$\frac{\partial g^{\alpha\beta}}{\partial \tau^1} = \eta^{\alpha\beta}.$$

One remaining question is to calculate the Weingarten operators for the submanifold using this second metric.

5. Frobenius submanifolds and the foldings of Coxeter graphs

In this section, the above ideas will be applied to a class of Frobenius manifolds constructed from a Coxeter group W and in particular two-dimensional Frobenius submanifolds will be considered.

The full details of the construction of these Frobenius manifolds may be found in [2]. For these, the Euler vector field takes the form

$$E=\sum_{i=1}^{m} d_i t^i \frac{\partial}{\partial t^i},$$

where d_i are the exponents of the Coxeter group, or equivalently, the degrees of the basic W-invariant polynomials. These are given in Table 1. (Note the reverse ordering, so $d_n = 2$, $d_1 = h$.) They satisfy the basic condition $d_i + d_{m+1-i} = h + 2$, where h is known as the Coxeter number of the group. The corresponding prepotential is polynomial, and it has been conjectured that all such polynomial prepotentials arise from this construction.

Using the parametrization (8) together with the requirement that the induced metric must be both flat and in flat coordinates implies that the two-dimensional submanifolds are parametrized:

$$t^1=\tau_1-\frac{1}{2}\int\sum_{j=2}^{m-1} f_j'(\tau_2) f_{m+1-j}'(\tau_2)\,\mathrm{d}\tau_2,$$
$$t^j = f_j(\tau_2),\quad j=2,\dots,m-1,\qquad t^m=\tau_2.$$

If the condition $(E|_{\mathcal{N}})^{\perp}=0$ is now imposed, one obtains simple equations for the f_i giving the parametrization

$$t^1=\tau_1-\frac{1}{4}\left\{\sum_{j=2}^{m-1} k_j k_{m+1-j} d_j d_{m+1-j}\right\}\frac{1}{h}\tau_2^{h/2},$$
$$t^j=k_j\tau_2^{d_j/2},\quad j=2,\dots,m-1,\qquad t^m=\tau_2$$

Table 1
Degrees of the W-invariant polynomials

Coxeter group	Exponents $d_n,\dots,d_l = h$
A_n	$2, 3, \dots, n+1$
B_n	$2, 4, 6, \dots, 2n$
D_n	$2, 4, 6, \dots, 2n-2, n$
E_6	2, 5, 6, 8, 9, 12
E_7	2, 6, 8, 10, 12, 14, 18
E_8	2, 8, 12, 14, 18, 20, 24, 30
F_4	2, 6, 8, 12
G_2	2, 6
H_3	2, 6, 10
H_4	2, 12, 20, 30
$I_2(m)$	$2, m$

(using the fact that $d_m = 2$ for all Coxeter groups, remembering the reverse ordering of the exponents) and the induced Euler vector field

$$E_{\mathcal{N}} = h\tau^1 \frac{\partial}{\partial \tau^1} + 2\tau^2 \frac{\partial}{\partial \tau^2}.$$

By Proposition 2, this submanifold automatically is a Frobenius (but not necessarily natural) submanifold and it is easy to check that the induced prepotential is

$$F_{\mathcal{N}} = \tfrac{1}{2}\tau_1^2\tau_2 + p(k_i)\tau_2^{h+1},$$

where $p(k_i)$ is some function of the constants k_i which define the submanifold. This prepotential is polynomial and corresponds to the Coxeter group $I_2(h)$. Thus for any Coxeter group, one has a family of two-dimensional Frobenius submanifold:

$$\mathcal{F}_W \to \mathcal{F}_{I_2(h)}.$$

Natural Frobenius manifolds occur at special values of the constants k_i.

Example. Consider the Frobenius manifold defined by the polynomial prepotential

$$F_{H_3} = \tfrac{1}{2}t_1^2t_3 + \tfrac{1}{2}t_1t_2^2 + \tfrac{1}{60}t_2^3t_3^2 + \tfrac{1}{20}t_2^2t_3^5 + \tfrac{1}{3960}t_3^{11},$$

and Euler vector field

$$E = 10t_1\frac{\partial}{\partial t_1} + 6t_2\frac{\partial}{\partial t_2} + 2t_3\frac{\partial}{\partial t_3}.$$

Such a manifold is associated to the Coxeter group H_3.

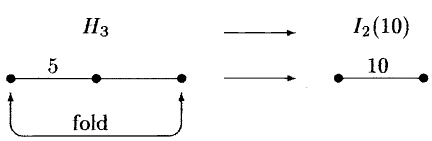
By Proposition 2, any submanifold $\mathcal{N}$ defined by

$$t_1 = \tau_1 - \tfrac{9}{10}k_2^2\tau_2^5, \qquad t_2 = k_2\tau_2^3, \qquad t_3 = \tau_2$$

is a Frobenius submanifold with respect to the induced structures. The condition for the manifold to be a natural Frobenius submanifold — normally a first order ordinary differential equation of degree 3 — reduces to a cubic polynomial

$$k_2(k_2 - 1)(27k_2 + 5) = 0.$$

Thus, there are three natural Frobenius submanifolds of this form. This Frobenius submanifold is also associated to a Coxeter group, namely $I_2(10)$. The relation between these two Coxeter groups may be seen in terms of the folding of their Coxeter diagrams:



where such folding preserves the Coxeter number (in this case 10) of the groups involved. When $k_2 = 0$, the submanifold is just a plane, and for only this value of k_2 does

$$F_{\mathcal{N}} = F|_{\mathcal{N}}.$$

Similar results have been obtained by Zuber [12] for natural Frobenius submanifolds obtained by foldings of arbitrary Coxeter diagrams, but the only submanifolds that were considered were hyperplanes. There are two other three-dimensional Coxeter groups, namely A_3 and B_3.

Example ($A_3 \to I_2(4)$). The prepotential for the Frobenius manifold constructed from A_3 is

$$F_{A_3} = \tfrac{1}{2}t_1^2t_3 + \tfrac{1}{2}t_1t_2^2 + \tfrac{1}{4}t_2^2t_3^2 + \tfrac{1}{60}t_3^5.$$

The two-dimensional submanifold is given by

$$t_1 = \tau_1 - \tfrac{9}{16}k_2^2\tau_2^2, \qquad t_2 = k_2\tau_2^{3/2}, \qquad t_3 = \tau_2.$$

The condition required for the submanifold to be a natural Frobenius submanifold reduces to $k_2(32 - 27k_2^2) = 0$. Thus, there are two natural Frobenius submanifolds given by $k_2 = 0, \pm\sqrt{32/27}$, i.e. the plane $t_2 = 0$ and the cylinder over the semi-cubical parabola $27t_2^2 = 32t_3^3$.

Example ($B_3 \to I_2(6)$). The prepotential for the Frobenius manifold constructed from B_3 is

$$F_{B_3} = \tfrac{1}{2}t_1^2t_3 + \tfrac{1}{2}t_1t_2^2 + \tfrac{1}{6}t_2^3t_3 + \tfrac{1}{6}t_2^2t_3^3 + \tfrac{1}{210}t_3^7.$$

The two-dimensional submanifold is given by

$$t_1 = \tau_1 - \tfrac{2}{3}k_2^2\tau_2^3, \qquad t_2 = k_2\tau_2^2, \qquad t_3 = \tau_2.$$

The condition required for the submanifold to be a natural Frobenius submanifold reduces to $k_2(2k_2 - 3)(-2k_2 - 1) = 0$. Thus, there are three natural Frobenius submanifolds given by $k_2 = 0, -\frac{1}{2}, +\frac{3}{2}$.

In the above three examples, the natural submanifolds are special from the point of view of singularity theory, the submanifolds are cylinders over the caustics of A_3, B_3 and H_3. This observation does not generalize directly, e.g. the cylinder over the caustic of A_4 is not a flat submanifold, and so cannot be a Frobenius submanifold. However, the induced multiplication is associative and quasi-homogeneous (since $(E_{\mathcal{N}})^{\perp} = 0$). These properties are best understood in terms of weak Frobenius and F-manifolds [6,7].

Example ($F_4 \to I_2(12)$). As a higher dimensional example, consider the embeddings of $I_2(12)$ in F_4. The prepotential for the Frobenius manifold constructed from F_4 is

$$F_{F_4} = \tfrac{1}{2}t_1^2t_4 + t_1t_2t_3 + \tfrac{1}{6}t_2^3t_4 + \tfrac{1}{12}t_3^4t_4 + \tfrac{1}{6}t_2t_3^2t_4^3 + \tfrac{1}{60}t_2^2t_4^5 + \tfrac{1}{252}t_3^2t_4^7 + \tfrac{1}{185328}t_4^{13}.$$

The two-dimensional submanifold is given by

$$t_1 = \tau_1 - 2k_2k_3\tau_2^6, \qquad t_2 = k_2\tau_2^4, \qquad t_3 = k_3\tau_2^3, \qquad t_4 = \tau_2.$$

The conditions required for the submanifold to be a natural Frobenius submanifold are

$$k_2 + 12k_2^2 + 5k_3^2 - 36k_2k_3^2 = 0, \qquad k_3(1 + 36k_2 - 144k_2^2 + 36k_3^2) = 0.$$

These algebraic equations are easily solved giving six two-dimensional natural Frobenius submanifolds (ignoring one complex solution):

$$(k_2, k_3) = \begin{cases} (0,0), & (-\frac{1}{12}, 0), \\ (-\frac{1}{36}, +\frac{1}{18}), & (+\frac{5}{12}, +\frac{1}{2}), \\ (-\frac{1}{36}, -\frac{1}{18}), & (+\frac{5}{12}, -\frac{1}{2}). \end{cases}$$

Further examples may easily be constructed using the known formulae for prepotentials constructed from Coxeter groups [12].

6. The quantum cohomology of $\mathbb{CP}^2$

The quantum cohomology of $\mathbb{CP}^2$ is given in terms of the prepotential

$$F = \frac{1}{2}t_1^2 t_3 + \frac{1}{2}t_1 t_2^2 + \sum_{n=1}^{\infty} \frac{N_n^{(0)} t_3^{3n-1}\, \mathrm{e}^{nt_2}}{(3n-1)!}$$

with

$$E = t_1 \frac{\partial}{\partial t_1} + 3\frac{\partial}{\partial t_2} - t_3 \frac{\partial}{\partial t_3},$$

where $N_n^{(0)}$ is the number of rational curves of degree n through $3n-1$ generic points. The equations of associativity imply the recursion relation

$$N_n^{(0)} = \sum_{i+j=n} \left[\binom{3n-4}{3i-2} i^2 j^2 - i^3 j \binom{3n-4}{3i-1} \right] N_i^{(0)} N_j^{(0)},$$

first derived by Kontsevich and Manin. With the initial condition $N_1^{(0)} = 1$, this determines all the $N_n^{(0)}$. Following the derivation in [2], this prepotential may be written as

$$F = \tfrac{1}{2}t_1^2 t_3 + \tfrac{1}{2}t_1 t_2^2 + t_3^{-1}\phi(x),$$

where $x = t_2 + 3\log t_3$. The equations of associativity then reduce to the third order ordinary differential equation

$$9\phi''' - 18\phi'' + 11\phi' - 2\phi = \phi''\phi''' - \tfrac{2}{3}\phi'\phi'' + \tfrac{1}{3}\phi''^2, \tag{13}$$

and with the ansatz

$$\phi(x) = \sum_{n=1}^{\infty} \frac{N_n^{(0)}}{(3n-1)!}\, \mathrm{e}^{nx}, \tag{14}$$

one obtains the above recursion relation.

By Proposition 2, any suitable two-dimensional submanifold is a Frobenius manifold, but a particularly interesting submanifold is given by $x = x_0$, where x_0 is a constant. On such a submanifold $(E|_{\mathcal{N}})^{\perp} = 0$. In terms of the parametrization,

$$t_1 = \tau_1 + \tfrac{9}{2}\tau_2^{-1}, \qquad t_2 = x_0 - 3\log\tau_2, \qquad t_3 = \tau_2,$$

one obtains a Frobenius manifold on $\mathcal{N}$ defined by

$$F_{\mathcal{N}} = \frac{1}{2}\tau_1^2\tau_2 - \left[\frac{81 - 8\phi(x_0) + 20\phi'(x_0)}{8}\right]\tau_2^{-1}, \qquad E_{\mathcal{N}} = \tau_1\frac{\partial}{\partial\tau_1} - \tau_2\frac{\partial}{\partial\tau_2}.$$

The obstruction to this being a natural Frobenius submanifold is

$$27 + 2\phi'(x_0) - 3\phi''(x_0) = 0. \tag{15}$$

It is not immediately obvious that a natural submanifold exists.

Lemma 4. *There exists a natural Frobenius submanifold, given by the condition $x = x_0$, where x_0 is the radius of convergence of the series* (14).

Proof. It was shown in [5] that the series (14) has a finite radius of convergence x_0. Moreover, it was shown that $\phi, \phi', \phi'', \phi'''$ are all positive with $\phi < \phi' < \phi'' < \phi'''$ for real $x < x_0$, and that ϕ, ϕ' and ϕ'' remains finite at x_0 with ϕ''' blowing up. Using these results, in the vicinity of x_0, ϕ takes the form

$$\phi = \phi_0 + \phi_1(x_0 - x) + \phi_2[\tfrac{1}{2}(x_0 - x)^2] + \lambda(x_0 - x)^{\alpha+2} + \cdots,$$

and substituting this into the differential equation (13) and equating coefficients yields $\alpha = \frac{1}{2}$, λ and the relation (15). Hence, natural Frobenius submanifolds exist. □

7. The quantum cohomology of $\mathbb{CP}^1 \times \mathbb{CP}^1$

As explained elsewhere [5], the quantum cohomology of $\mathbb{CP}^1 \times \mathbb{CP}^1$ is given in terms of the prepotential

$$F = \frac{1}{2}t_1^2 t_4 + t_1 t_2 t_3 + \sum_{\substack{a,b\geq 0\\ a+b\geq 1}} \frac{N_{ab}}{[2(a+b)-1]!} t_4^{2(a+b)-1}\, \mathrm{e}^{at_2 + bt_3},$$

and Euler vector field

$$E = \frac{1}{2}t_1\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} + \frac{\partial}{\partial t_3} - \frac{1}{2}t_4\frac{\partial}{\partial t_4}.$$

The coefficients N_{ab} are the number of rational curves on a smooth quadric (such quadrics being isomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$) with bidegree (a, b) through $2(a + b) - 1$ points. These are determined by the initial conditions $N_{01} = 1$, $N_{ab} = N_{ba}$ and the recursion relations

$$2abN_{ab}=\sum N_{a_1b_1}N_{a_2b_2}a_1^2b_2^2(a_1b_2-a_2b_1)\begin{pmatrix}2(a+b)-2\\2(a_1+b_1)-1\end{pmatrix},$$

$$aN_{ab}=\sum N_{a_1b_1}N_{a_2b_2}a_1(a_1^2b_2^2-a_2^2b_1^2)\begin{pmatrix}2(a+b)-3\\2(a_1+b_1)-1\end{pmatrix},$$

$$0=\sum N_{a_1b_1}N_{a_2b_2}a_1^2[(a_2+b_2-1)(b_1a_2+b_2a_1)-a_2b_2(2(a_1+b_1)-1)]\times\begin{pmatrix}2(a+b)-3\\2(a_1+b_1)-1\end{pmatrix},\qquad N_{ab}=\sum N_{a_1b_1}N_{a_2b_2}(a_1b_2+a_2b_1)b_2\times\left[a_1\begin{pmatrix}2(a+b)-4\\2(a_1+b_1)-2\end{pmatrix}-a_2\begin{pmatrix}2(a+b)-4\\2(a_1+b_1)-3\end{pmatrix}\right],$$

where the sums are over $a_1, a_2, b_1, b_2 \geq 0$, $a_1+a_2=a$, $b_1+b_2=b$.

The symmetry $t_2 \leftrightarrow t_3$ in these formulae suggest that one should consider the codimension 1 submanifold defined by the parametrization

$$t_1=\tau_1,\qquad t_2=\frac{1}{\sqrt{2}}\tau_2,\qquad t_3=\frac{1}{\sqrt{2}}\tau_2,\qquad t_4=\tau_3,$$

where the factor $\sqrt{2}$ ensures that the induced metric takes the canonical antidiagonal form. This submanifold also satisfies the condition $(E|_{\mathcal{N}})^{\perp}=0$ so that

$$E_{\mathcal{N}}=\frac{1}{2}\tau_1\frac{\partial}{\partial\tau_1}+\sqrt{2}\frac{\partial}{\partial\tau_2}-\frac{1}{2}\tau_3\frac{\partial}{\partial\tau_3}.$$

The calculation of the induced multiplication on $\mathcal{N}$ is particularly simple, due to the fact that $\mathcal{N}$ is just a hyperplane. The induced structure $\mathcal{F}_{\mathcal{N}}$ is a natural Frobenius submanifold, the obstructions all take the form

$$\Xi=\sum(a-b)S(a,b)$$

with $S(a,b)=S(b,a)$ and hence vanish. The induced prepotential is given by

$$F_{\mathcal{N}}=F|_{\mathcal{N}}=\frac{1}{2}\tau_1^2\tau_3+\frac{1}{2}\tau_1\tau_2^2+\tau_3^{-1}\sum_{n=1}^{\infty}\frac{\left[\sum_{r=0}^{n}N_{n-r,r}\right]}{(2n-1)!}\tau_3^{2n}\,\mathrm{e}^{n\tau_2/\sqrt{2}}.$$

While this construction guarantees that $\mathcal{F}_{\mathcal{N}}$ is a Frobenius manifold, it is interesting to calculate the relations required to ensure that the prepotential

$$F=\frac{1}{2}\tau_1^2\tau_3+\frac{1}{2}\tau_1\tau_2^2+\tau_3^{-1}\sum_{n=1}^{\infty}\frac{N_n}{(2n-1)!}\tau_3^{2n}\,\mathrm{e}^{n\tau_2/\sqrt{2}}$$

defines a Frobenius manifold. The calculations are identical, apart from different numerical coefficients, to the calculation of the quantum cohomology of $\mathbb{P}^2$, so the details will not be

Table 2
The numbers N_n for $1 \leq n \leq 12$

n	$N_n = \sum_{r=0}^{n} N_{n-r,r}$
1	2
2	1
3	2
4	14
5	194
6	4792
7	182770
8	10078480
9	758120642
10	74795167616
11	937456239394
12	1456089241205248

repeated. It turns out that the coefficients N_n must satisfy the recursion relation

$$N_n = \frac{1}{2}(2n-4)! \sum_{\substack{k\geq 1, l\geq 1 \\ k+l=n}} \frac{kl[kl(n+1)-(l^2+k^2)]}{(2k-1)!(2l-1)!} N_k N_l$$

with initial condition $N_2 = 2$ (see Table 2). Thus, the numbers $N_n = \sum_{r=0}^{n} N_{n-r,r}$ must satisfy the above recursion relation. This may be easily verified for small values of n, but the fact that $\mathcal{N}$ is a *natural* Frobenius submanifold makes the result automatic. Presumably, one may also derive this result directly from the recursion relations. Obviously, the numbers N_n contain less information than the original N_{ab}, the Frobenius submanifold only determining their sum, not the individual numbers.

One may also, mirroring the construction in Section 8, obtain a Frobenius submanifold of $\mathcal{F}_{\mathcal{N}}$ on the submanifold of $\mathcal{N}$ defined by the condition

$$\frac{1}{\sqrt{2}}\tau_2 + 2\log\tau_3 = \text{constant}.$$

Thus, one obtains a nested sequence of Frobenius manifolds.

Underlying this construction is the symmetry $t_2 \leftrightarrow t_3$. The origin of this symmetry comes from the fact that the Frobenius manifold is a tensor product of 2 two-dimensional Frobenius manifolds [8],

$$\mathcal{F}_{\mathbb{CP}^1\times\mathbb{CP}^1} \cong \mathcal{F}_{\mathbb{CP}^1} \otimes \mathcal{F}_{\mathbb{CP}^1}, \tag{16}$$

where $\mathcal{F}_{\mathbb{CP}^1}$ is given by

$$\mathcal{F}_{\mathbb{CP}^1} = \frac{1}{2}t_1 t_2^2 + \mathrm{e}^{t_2}, \qquad E_{\mathbb{CP}^1} = \frac{1}{2}t_1\frac{\partial}{\partial t_1} + 2\frac{\partial}{\partial t_2}.$$

The Euler vector field for the product (16), constructed from $E_{\mathbb{CP}^1}$, is

$$E_{\mathbb{CP}^1\times\mathbb{CP}^1} = t^{11}\frac{\partial}{\partial t^{11}} + 2t^{12}\frac{\partial}{\partial t^{12}} + 2t^{21}\frac{\partial}{\partial t^{21}} - t^{22}\frac{\partial}{\partial t^{22}},$$

and this, by construction, automatically has the required symmetry $t^{12} \leftrightarrow t^{21}$. Thus, the natural Frobenius submanifold may be formulated in terms of a quotient of this product by this symmetry:

$$\mathcal{F}_{\mathcal{N}} \cong \frac{\mathcal{F}_{\mathbb{CP}^1} \times \mathcal{F}_{\mathbb{CP}^1}}{t^{12} \leftrightarrow t^{21}}.$$

More generally, one may obtain new Frobenius manifolds by squaring a Frobenius manifold and taking such a quotient

$$\mathcal{F}_{\mathcal{N}} \cong \frac{\mathcal{F}_{\mathcal{M}} \otimes \mathcal{F}_{\mathcal{M}}}{\sim}.$$

Example. Another example of this kind is given in terms of the Frobenius manifold

$$F_{A_2} = \tfrac{1}{2} t_1^2 t_2 + t_2^4,$$

which is constructed from the Coxeter group $A_2 \cong I_2(3)$. The product of two such manifolds is again a Frobenius manifold associated to the Coxeter group D_4:

$$\mathcal{F}_{D_4} \cong \mathcal{F}_{A_2} \otimes \mathcal{F}_{A_2}.$$

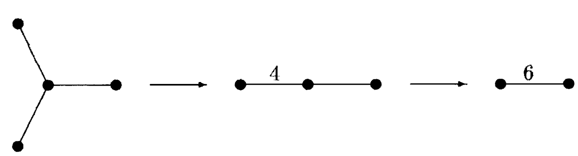
By construction, this automatically has the symmetry $t^{12} \leftrightarrow t^{21}$ so, as in the case of $\mathcal{F}_{\mathbb{CP}^1} \otimes \mathcal{F}_{\mathbb{CP}^1}$, one has a natural Frobenius submanifold defined on the hyperplane $t^{12} = t^{21}$. This Frobenius submanifold is again associated to a Coxeter group, namely B_3:

$$\mathcal{F}_{B_3} \cong \frac{\mathcal{F}_{A_2} \otimes \mathcal{F}_{A_2}}{\sim}.$$

Repeating the construction outlined in Section 1, one obtains natural Frobenius submanifolds inside $\mathcal{F}_{B_3}$, this time associated to the Coxeter group $I_2(6)$. Thus, one obtains a nested sequence of natural Frobenius manifolds

$$\mathcal{F}_{I_2(6)} \subset \mathcal{F}_{B_3} \subset \mathcal{F}_{A_2} \otimes \mathcal{F}_{A_2} \cong \mathcal{F}_{D_4}.$$

This sequence may be understood in terms of foldings of Coxeter diagrams:



One may also embed the trivial one-dimensional Frobenius manifold given by $F = \frac{1}{6} t_1^3$ in $F_{I_2(6)}$, giving complete nested sequence of Frobenius submanifolds.

8. Conclusion

The results of this paper have been derived using flat coordinates only. One important avenue for future research is to rederive them using canonical coordinates. Such an approach will involve the classical differential geometric problem of properties of flat submanifolds of Ergoff metrics which are themselves Ergoff. One basic object, that is best studies using canonical coordinates, is the isomonodromic τ-function, denoted by $\tau_{\rm I}$. One obvious question is how the $\tau_{\rm I}$-function of a (natural) submanifold is related to that of its parent Frobenius manifold. As the following discussion will show, the relation, whatever it is, is not straightforward.

One way to study certain properties of the $\tau_{\rm I}$-function without having to use canonical coordinates is to use the following result

$$\tau_{\rm I} = J^{1/24}\,{\rm e}^{G} \tag{17}$$

recently proved in [3] for semi-simple Frobenius manifolds. Here J is the Jacobian of the transformation from canonical to flat coordinates, and G is the solution to Getzler's equations for genus-1 Gromov–Witten invariants. Consider the Frobenius manifolds constructed from the Coxeter groups A_3, B_3 and H_3. The corresponding G-functions are

$$G_{A_3} = 0, \qquad G_{B_3} = -\tfrac{1}{48}\log[2t_2 - 3t_3^2], \qquad G_{H_3} = -\tfrac{1}{20}\log[t_2 - t_3^3].$$

In the latter two cases, G has a logarithmic singularity on *one* of the corresponding natural Frobenius submanifolds. In all cases, these natural submanifolds lie in the nilpotent locus, so from (17) the $\tau_{\rm I}$-function is singular on all three of the natural Frobenius submanifolds. This property is also present in the Frobenius manifold for the quantum cohomology of $\mathbb{CP}^2$. The derivative of the G-function is given by

$$G' = \frac{\phi''' - 27}{8(27 + 2\phi' - 3\phi'')},$$

where ϕ is given by (13), and using the series expansion (14) one may integrate this equation and show that G also has a logarithmic singularity on the natural Frobenius submanifold. This submanifold does not lie in the nilpotent locus, and it follows from (17) that $\tau_{\rm I}$ is also singular on the submanifold.

For the quantum cohomology of $\mathbb{CP}^1 \times \mathbb{CP}^1$, it is unclear what the singularity structure of the G-functions is since its governing equations are somewhat more complicated, but even if the G-function does restrict to the submanifolds, it does not restrict to the G-function of the submanifold. This is easily seen by calculating the scaling constant γ defined by $\mathcal{L}_E G = \gamma$. The scaling constant of the $G|_{\mathcal{N}}$ does not equal the scaling constant of $G_{\mathcal{N}}$.

It has been shown that the contracted genus-0 Gromov–Witten invariants $\sum_{a+b=n} N_{ab}^{(0)}$ satisfy a simple recursion relation which may be understood as coming from a natural codimension 1 Frobenius submanifold. This raises the question of how higher-genus contracted Gromov–Witten invariants $\sum_{a+b=n} N_{ab}^{(g)}$ are related, if at all, to this submanifold. It would also be of interest both to have a direct proof of the genus-0 result by contracting the full recursion relations for $N_{ab}^{(0)}$, and to have a direct algebraic–geometric proof of why this submanifold 'counts' these contracted sums.

In summary, the results suggest the following problems:

- How can one reformulate these results in terms of canonical coordinates?

- How is the singularity structure of the G-function related to the existence of natural Frobenius submanifolds?
- If $\mathcal{N} \subset \mathcal{M}$ is a natural Frobenius submanifold, what are the relationships

  $$(\tau_{\mathrm{I}})_{\mathcal{N}} \leftrightarrow (\tau_{\mathrm{I}})_{\mathcal{M}}, \qquad G_{\mathcal{N}} \leftrightarrow G_{\mathcal{M}}?$$

  These are clearly related by (17). For the KP hierarchy there are some interesting results [1] on the Birkhoff strata of the Grassmannian based on the zeros of the τ-function. It would be interesting to study the dispersionless counterparts of such systems.

Finally, it should be possible to study degenerate Frobenius manifolds introduced in [11] in this framework, by embedding them in higher-dimensional, non-degenerate Frobenius manifolds [9].

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I would like to thank Claus Hertling for his comments on this work, and in particular for pointing out the relation between caustics and Frobenius submanifolds.

Appendix A

Since a Frobenius manifold is flat, any Frobenius submanifold must also be flat, and hence one has to consider the possible embedding of one flat space in another. The following results are entirely standard (see, e.g. [4]) and are just a specialization of the general Gauss–Codazzi equations for the embedding of an arbitrary manifold into another.

From (1), the induced metric on $\mathcal{N}$ is

$$\eta_{\alpha\beta} = \eta_{ij}\frac{\partial t^i}{\partial\tau^\alpha}\frac{\partial t^j}{\partial\tau^\beta}. \tag{A.1}$$

It will be assumed that the τ^α-coordinates are flat coordinates, i.e. the components of $\eta_{\alpha\beta}$ are constant. Let $n_{\tilde\alpha}^j$ be a field of normal vectors to $\mathcal{N}$, so

$$\eta_{ij}\frac{\partial t^i}{\partial\tau^\alpha}n_{\tilde\alpha}^j = 0, \tag{A.2}$$

normalized so that

$$\eta_{ij}n_{\tilde\alpha}^i n_{\tilde\beta}^j = \eta_{\tilde\alpha\tilde\beta}, \tag{A.3}$$

where $\eta_{\tilde\alpha\tilde\beta}$ are constant with $\eta_{\tilde\alpha\tilde\beta} = \epsilon(\tilde\alpha)\delta_{\tilde\alpha\tilde\beta}$ with $\epsilon(\tilde\alpha) = \pm 1$.

Differentiating (A.1) implies

$$\eta_{ij}\frac{\partial^2 t^i}{\partial\tau^\alpha\partial\tau^\beta}\frac{\partial t^j}{\partial\tau^\beta} = 0,$$

and hence there exist functions $\Omega_{\alpha\beta}{}^{\tilde\alpha}$ such that

$$\frac{\partial^2 t^i}{\partial\tau^\alpha\partial\tau^\beta} = \Omega_{\alpha\beta}{}^{\tilde\alpha}n_{\tilde\alpha}{}^i. \tag{A.4}$$

Differentiating (A.2) implies, on using (A.4)

$$\Omega_{\tilde{\alpha}\alpha\beta} = -\eta_{ij}\frac{\partial t^i}{\partial \tau^\alpha}\frac{\partial n_{\tilde{\alpha}}{}^j}{\partial \tau^\beta}. \tag{A.5}$$

Differentiating (A.3) implies that

$$\frac{\partial n^j_{\tilde{\beta}}}{\partial \tau^\alpha} = -\Omega_{\tilde{\beta}\alpha\sigma}\frac{\partial t^j}{\partial \tau^\nu}\eta^{\sigma\mu}. \tag{A.6}$$

Note, in particular, that the torsion tensors are zero. The immediate consequence of this is that the normal bundle of $\mathcal{N}$ is flat, i.e. $d\vec{n}_\alpha \in T\mathcal{N}$.

The Gauss–Codazzi equations, the integrability conditions for the above structures, reduce to the three equations

$$\eta^{\tilde{\alpha}\tilde{\beta}}[\Omega_{\tilde{\alpha}\alpha\beta}\Omega_{\tilde{\beta}\gamma\delta} - \Omega_{\tilde{\alpha}\alpha\delta}\Omega_{\tilde{\beta}\gamma\beta}] = 0, \tag{A.7}$$

$$\eta^{\mu\nu}[\Omega_{\tilde{\alpha}\mu\alpha}\Omega_{\tilde{\beta}\nu\beta} - \Omega_{\tilde{\alpha}\mu\beta}\Omega_{\tilde{\beta}\nu\alpha}] = 0, \tag{A.8}$$

$$\frac{\partial \Omega_{\tilde{\alpha}\alpha\mu}}{\partial \tau^\nu} - \frac{\partial \Omega_{\tilde{\alpha}\alpha\nu}}{\partial \tau^\mu} = 0. \tag{A.9}$$

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